

Linear relations of zeroes of the zeta-function

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Abstract

This article considers linear relations between the non-trivial zeroes of the Riemann zeta-function. The main application is an alternative disproof to Mertens' conjecture by showing that $\limsup_{x \rightarrow \infty} M(x)x^{-1/2} \geq 1.6383$, and $\liminf_{x \rightarrow \infty} M(x)x^{-1/2} \leq -1.6383$.

1 Introduction and Results

It is not known whether any non-trivial zeroes of the zeta-function are linearly dependent over the rationals. That is, no one has found an $N \geq 1$ and integers c_1, \dots, c_N , not all zero, for which

$$\sum_{n=1}^N c_n \gamma_n = 0, \quad (1.1)$$

where $\rho_n = \beta_n + i\gamma_n$ is the n th non-trivial zero of $\zeta(s)$. It seems that Ingham [9] was the first to consider (1.1). His paper concerned, *inter alia*, Mertens' conjecture that $M(x) = \sum_{n \leq x} \mu(n) \leq x^{\frac{1}{2}}$, where $\mu(n)$ is the Möbius function. Ingham showed that Mertens' conjecture implies that there are infinitely many linear dependencies as given in (1.1). Since there seems to be no intrinsic reason why (1.1) should be true, Ingham expressed doubts about Mertens' conjecture. Indeed, Mertens' conjecture was shown to be false by Odlyzko and te Riele in [12].

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Ingham's result is 'doubly infinite': there are infinitely many choices for the c_n and infinitely many N . Bateman et al. [2] proved a 'singly infinite' result: Mertens' conjecture implies that there are infinitely many sums of the type (1.1) in which the c_n are integers, not all zero, $|c_n| \leq 2$, and at most one $c_n = \pm 2$. Furthermore, they considered all of these permissible sums for $1 \leq N \leq 20$ and showed that no linear dependencies exist. We extend their table in Appendix A.

The contrapositive to the statement given by Bateman et al. is an interesting one: if there is no relation of the type (1.1) for $|c_n| \leq 2$, then Mertens' conjecture is false. This singly infinite result was reduced to a finite result with the work of Grosswald [6]. Following Grosswald, we are able to prove the following

Theorem 1.

$$\limsup_{x \rightarrow \infty} M(x)x^{-1/2} \geq 1.6383, \quad \liminf_{x \rightarrow \infty} M(x)x^{-1/2} \leq -1.6383.$$

This improves on the result of Kotnik and te Riele [10] who showed that $\limsup_{x \rightarrow \infty} M(x)x^{-1/2} \geq 1.218$ and $\liminf_{x \rightarrow \infty} M(x)x^{-1/2} \leq -1.229$. An added feature to the approach in this paper is that the bounds given in Theorem 1 for the \limsup and the \liminf are equal.

2 Outline

Suppose $g(x)$ is a piecewise-continuous function that is bounded on finite intervals. Suppose also that

$$G(s) = \int_1^\infty g(x)x^{-s-1} dx \quad (2.1)$$

originally absolutely convergent for $\sigma > \sigma_a$, say, can be continued analytically to $\sigma = \sigma_0$. Moreover, assume that one can write the principal part of $G(s)$ as

$$H(s) = G(s) - \left(\frac{r_0}{s - \sigma_0} + \sum_\gamma \frac{r_\gamma}{s - (\sigma_0 + i\gamma)} \right), \quad (2.2)$$

where $H(s)$ is analytic for $s = \sigma_0 + it$, where $|t| < T$. Here, γ is an element of some finite set of numbers $0 < |\gamma| < T$. Ingham [9, Thm 1] proved that for any $T > 0$ and any x_0 ,

$$\liminf_{x \rightarrow \infty} \frac{g(x)}{x^{\sigma_0}} \leq r_0 + \sum_{-T < \gamma < T} r_\gamma \left(1 - \frac{|\gamma|}{T} \right) x_0^{i\gamma} \leq \limsup_{x \rightarrow \infty} \frac{g(x)}{x^{\sigma_0}}. \quad (2.3)$$

Ingham actually proved a slightly different version from that given above — see [7] and [1, p. 86] for details. From (2.3), Ingham was able to prove that if the γ s were linearly independent, then

$$\liminf_{x \rightarrow \infty} \frac{M(x)}{\sqrt{x}} = -\infty, \quad \limsup_{x \rightarrow \infty} \frac{M(x)}{\sqrt{x}} = \infty. \quad (2.4)$$

This is achieved using Kronecker's theorem — see [9, pp. 318-319]. Ingham also gave a proof of the classical result that (2.4) follows if either the Riemann hypothesis is false or not all the zeroes are simple.

Grosswald's idea is to obtain only finite upper and lower bounds in (2.4) by weakening the hypothesis that the γ s are linearly independent. This weaker version of linear independence has also been considered by Anderson and Stark [1] and Diamond [5]. The following version is that given in [1].

Let $T > 0$ be given. Let Γ denote the set of positive ordinates of zeroes of the Riemann zeta-function. Define Γ' as a subset of Γ such that every $\gamma \in \Gamma'$ lies in the range $0 < \gamma < T$. Finally, let $\{N_\gamma\}$ be a set of positive integers defined for $\gamma \in \Gamma'$.

Definition 1. *The elements of Γ' are N_γ -independent in $\Gamma \cap [0, T]$ if*

$$\sum_{\gamma \in \Gamma'} c_\gamma \gamma = 0, \quad \text{with } |c_\gamma| \leq N_\gamma, \quad (2.5)$$

implies that all $c_\gamma = 0$ and for any $\gamma^ \in \Gamma \cap [0, T]$,*

$$\sum_{\gamma \in \Gamma'} c_\gamma \gamma = \gamma^*, \quad \text{with } |c_\gamma| \leq N_\gamma, \quad (2.6)$$

implies that $\gamma^ \in \Gamma'$, that $c_{\gamma^*} = 1$, and that all other $c_\gamma = 0$.*

With this definition, it is possible to prove the following diluted version of (2.3).

Theorem 2 (Anderson and Stark). *If the elements of Γ' are N_γ -independent in $\Gamma \cap [0, T]$, then*

$$\liminf_{x \rightarrow \infty} \frac{g(x)}{x^{\sigma_0}} \leq r_0 - \sum_{\gamma \in \Gamma'} \frac{2N_\gamma}{N_\gamma + 1} |r_\gamma| \left(1 - \frac{|\gamma|}{T}\right) \quad (2.7)$$

and

$$\limsup_{x \rightarrow \infty} \frac{g(x)}{x^{\sigma_0}} \geq r_0 + \sum_{\gamma \in \Gamma'} \frac{2N_\gamma}{N_\gamma + 1} |r_\gamma| \left(1 - \frac{|\gamma|}{T}\right). \quad (2.8)$$

2.1 Mertens' Conjecture

One may assume the Riemann hypothesis and the simplicity of the zeroes, since otherwise (2.4) is true, and there is nothing else to show. In (2.1), take $g(x) = M(x)$ and so $\sigma_0 = \frac{1}{2}$ and, in (2.2), $r_0 = 0$ and $r_\gamma = (\rho \zeta'(\rho))^{-1}$. Let $M \geq 1$ be given. Choose $\Gamma' = \{\gamma_1, \dots, \gamma_M\}$ and choose $T = \gamma_{L+1} - \epsilon$ for some small ϵ , where $L \geq M$. Thus, $\Gamma \cap [0, T] = \{\gamma_1, \dots, \gamma_L\}$. Therefore, provided that (2.5) and (2.6) are satisfied, we have by Theorem 2

$$\liminf_{x \rightarrow \infty} \frac{M(x)}{x^{1/2}} \leq - \sum_{n=1}^M \frac{2N_{\gamma_n}}{N_{\gamma_n} + 1} \frac{1}{|\rho_n \zeta'(\rho_n)|} \left(1 - \frac{|\gamma_n|}{T}\right) \quad (2.9)$$

and

$$\limsup_{x \rightarrow \infty} \frac{M(x)}{x^{1/2}} \geq \sum_{n=1}^M \frac{2N_{\gamma_n}}{N_{\gamma_n} + 1} \frac{1}{|\rho_n \zeta'(\rho_n)|} \left(1 - \frac{|\gamma_n|}{T}\right). \quad (2.10)$$

2.2 Computation

Previous disproofs of Mertens' conjecture have utilized the basis reduction algorithm first described by Lenstra, Lenstra and Lovász in [11], called LLL-reduction. We also employ the use of this robust algorithm, but in a different way.

Definition 2. Let $K \in \mathbb{Z}, S = \{x_1, \dots, x_n\} \subset \mathbb{R}$ and define x'_i such that $Kx'_i = \lfloor Kx_i \rfloor$. Then $L(K; S) \subset \mathbb{Z}^{n+1}$ is the lattice generated by the following vectors

$$\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ Kx'_1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \\ Kx'_2 \end{pmatrix}, \dots, \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ Kx'_n \end{pmatrix}.$$

Any vector in $L(K; S)$ will be of the form:

$$a_1 \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ Kx'_1 \end{pmatrix} + a_2 \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \\ Kx'_2 \end{pmatrix} + \dots + a_n \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ Kx'_n \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \\ Kx \end{pmatrix},$$

where $x = a_1x'_1 + a_2x'_2 + \dots + a_nx'_n$. Using this lattice, will break the verification of (2.6) into two pieces. For this section, we will assume that each N_γ is the same. First, we loosen the restriction of (2.5) such that $|c_\gamma| \leq N_\gamma + 1$ with at most one $|c_\gamma| = N_\gamma + 1$. This accounts for all $\gamma^* \in \Gamma'$ in (2.6).

Lemma 1. If $S = \{\gamma_1, \gamma_2, \dots, \gamma_n\} \subset \mathbb{R}$ is linearly dependent (i.e., $\alpha_1\gamma_1 + \dots + \alpha_n\gamma_n = 0$ with $|\alpha_i| \leq N_\gamma + 1$ with at most one i such that $|\alpha_i| = N_\gamma + 1$), then there exists a nonzero vector $v \in L(K; S)$ such that $|v|^2 < (n^2 + n)N_\gamma^2 + (2n + 2)N_\gamma + 2$.

Proof. Consider the following vector $v \in L(K; S)$,

$$v = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \\ K(\alpha_1\gamma'_1 + \dots + \alpha_n\gamma'_n) \end{pmatrix}.$$

The assumptions of the lemma show

$$\begin{aligned} |v|^2 &= \alpha_i^2 + \cdots + \alpha_n^2 + K^2(\alpha_1\gamma'_1 + \cdots + \alpha_n\gamma'_n)^2 \\ &= \alpha_i^2 + \cdots + \alpha_n^2 + K^2\{(\alpha_1\gamma'_1 + \cdots + \alpha_n\gamma'_n) - (\alpha_1\gamma_1 + \cdots + \alpha_n\gamma_n)\}^2, \end{aligned}$$

since $\alpha_1\gamma_1 + \cdots + \alpha_n\gamma_n = 0$. Upon using the upper bounds on $|\alpha_i|$ and the fact that $|\gamma_i - \gamma'_i| \leq K^{-1}$, it follows that

$$|v|^2 \leq nN_\gamma^2 + 2N_\gamma + 1 + K^2 \left(\frac{nN_\gamma + 1}{K} \right)^2,$$

whence the lemma follows. \square

Similarly, to account for the remaining zeroes (i.e., $\gamma^* \notin \Gamma'$), we have the following

Lemma 2. *If $S = \{\gamma_1, \gamma_2, \dots, \gamma_n, \gamma_t\} \subset \mathbb{R}$ is linearly dependent (i.e., $\alpha_1\gamma_1 + \cdots + \alpha_n\gamma_n + \alpha_t\gamma_t = 0$ with $|\alpha_i| \leq N_\gamma$ for $1 \leq i \leq n$ and $|\alpha_t| \leq 1$), then there exists a nonzero vector $v \in L(K; S)$ such that $|v|^2 < (n^2 + n)N_\gamma^2 + 2nN_\gamma + 2$.*

Proof. The proof follows that of Lemma 1; for each i , we have $|\alpha_i| \leq N_\gamma$. \square

Note that in both lemmas above, the bounds are independent of our choice of K . The following lemma is true of all lattices.

Lemma 3. *[3, Proposition 3.14] Let $L \subset \mathbb{Z}^n$ be a lattice of dimension m . Then $|x|^2 \geq \min(|b_i^*|^2)$ for any nonzero $x \in L$. (Where $\{b_i^*\}$ is the Gram-Schmidt orthogonalization of the basis of L .)*

Lemmas 1, 2 and 3 may be summarized by the following

Theorem 3. *Let $L_0 = L(K; \Gamma')$ and let $L_t = L(K; \Gamma' \cup \{\gamma_t\})$ where $\gamma_t \in [0, T] \setminus \Gamma'$. We define $\{b_1, \dots, b_n\}$ and $\{\beta_{t,1}, \dots, \beta_{t,n}, \beta_{t,t}\}$ to be a basis for each lattice, respectively.*

The elements of Γ' are N_γ -independent in $\Gamma \cap [0, T]$ if

$$\min(|b_i^*|^2) \geq (n^2 + n)N_\gamma^2 + (2n + 2)N_\gamma + 2$$

and

$$\min(|\beta_{t,i}^*|^2) \geq (n^2 + n)N_\gamma^2 + 2nN_\gamma + 2$$

for all $\gamma_t \in [0, T] \setminus \Gamma'$.

Rearranging the equations above, we find that given a basis for our lattice, we may determine a lower bound for which N_γ may take.

A quick computation shows that the original bases of each lattice give us no information about the linear dependence of the elements of Γ' . For this reason, we must find alternative bases for each lattice. We apply the LLL-reduction algorithm to find a *nearly orthogonal basis* for each lattice.

We chose $K = 10^k$ for some positive k : the x'_i s are simply γ_i s accurate to k decimal places (and then truncated). We used GP/Pari and Sage to compute each γ_i very accurately.

Once the basis is reduced via LLL-reduction,

$$|b_k^*|^2 \geq |b_{k-1}^*|^2 \left(\delta - \frac{1}{4} \right)$$

for all permissible k . When computing the LLL-reduction, we tested several values of δ and k to determine whether there was a significant difference in the choices. It turns out that the choice of δ is far less important than the choice of k . We refer the reader to [4, Ch. 2] for a closer look at LLL-reduction, including basically the same set up of vectors to determine the linear dependence of a set of numbers.

We used $k = 9000$ and the standard $\delta = \frac{3}{4}$ when performing LLL-reduction on L_0 and a weaker $\delta = \frac{3}{10}$ when reducing each L_t to obtain the results mentioned in Theorem 1. To determining the value of T , we are limited only by the time to compute each zero and to perform the LLL-reduction on L_0 and L_t .

2.3 An improved kernel

In Theorem 2, Anderson and Stark follow Ingham and use the Féjer kernel

$$f(t) = \begin{cases} 1 - \frac{|t|}{T}, & |t| \leq T, \\ 0, & |t| > T \end{cases}$$

to truncate the relevant sums. What is really sought is a function which has a non-negative Fourier transform, is supported on $[-1, 1]$, and for which the function is as close as possible to unity in a neighbourhood about $t = 0$. The last condition ensures that the contributions of the lower-lying zeroes are maximised. We use the function

$$f_0(t) = \begin{cases} (1 - \frac{|t|}{T}) \cos \frac{\pi t}{T} + \pi^{-1} \sin \frac{\pi |t|}{T}, & |t| \leq T, \\ 0, & |t| > T \end{cases}$$

which is used by Odlyzko and te Riele in [12] — for a discussion about the origin of this function see [12, §4.1].

2.4 Sorting the zeroes

Though the indices mentioned above may suggest that we must use the first n zeroes as Γ' , this is not the case. Since we want to maximize equation (2.9), we shall sort the zeroes based on the ordering \prec defined as follows

$$\gamma_i \prec \gamma_j \iff \frac{1}{|\rho_i \zeta'(\rho_i)|} f_0(\gamma_i) > \frac{1}{|\rho_j \zeta'(\rho_j)|} f_0(\gamma_j).$$

For small values of i , sorting via \prec does not seem to affect the order very much. However, as T increases, the zeroes become scrambled. We sorted the zeroes $\gamma_1, \dots, \gamma_{2000}$ (using $T \approx \gamma_{2001} - \epsilon$) and set Γ' to be the first 500 zeroes in this list. After running the corresponding lattice through LLL-reduction as in Lemma 1, we find that the largest value N_γ may take is 794948. We now apply Lemma 2 to all $\gamma_t \in [0, T] \setminus \Gamma'$. After checking each of these, we arrive at the following

Theorem 4. *Let Γ' be the first 500 zeroes (in the sorted order above) with $T = \gamma_{2001} - \epsilon$. Then the elements of Γ' are 4976-independent.*

2.5 Improvements

Naturally, one should like to choose Γ' to have as many entries as possible and K to be as large as possible to improve on the bounds in Theorem 1. Unfortunately, the time taken to run each LLL-reduction is $O(n^6 \log^3 K) = O(n^6 k^3)$, so either one of these choices results in a quick computational explosion.

If we enlarge n without changing K , the value of N_γ is likely to decrease dramatically. Our experiments have shown that the value of N_γ tends to drop to zero eventually as n increases. On the flip-side, increasing the value of K when the value of N_γ is already large is relatively fruitless, as an increase of K can only increase N_γ and the $\frac{2N_\gamma}{N_\gamma+1}$ term is already close to 2.

One might also suggest taking a larger value of T , re-sorting the zeroes and computing the corresponding N_γ . This seems to be the best possibility. By re-sorting the zeroes for each pair of n and T , one has the *optimal* solution (provided the values of N_γ remains large). However, if one wishes to roll the dice with different values of n and T , one must do most of each computation from scratch. The first part of this recalculation can be cut down dramatically by storing specific intermediate results of the LLL-reduction and starting the reduction part way through. The second part of the recalculation, however, must be completely redone each time a new set of zeroes is selected.

Say we fix n and we wish to send $T \rightarrow \infty$, sorting the zeroes once again for each selection of T . In order for a high zero to have a large contribution, its derivative must be small. However, in [8], it is stated that *small* values of $|\zeta'(\frac{1}{2} + i\gamma)|$, about 0.002, do not appear until $|\gamma| \approx 10^{22}$, meaning that their contribution to the sum will be minuscule. Thus, once T is raised past a *reasonable* height it is unlikely that the first n sorted zeroes will change.

Figure 2.5 shows the relationship between n and T with resorting of the zeroes. The chart assumes that $\frac{2N_\gamma}{N_\gamma+1} \approx 2$, which is a fair assumption if one believes that the zeroes are indeed linearly independent. Of special note, sorting the initial 9000 zeroes and taking the *best* 1000 in sorted order gives the first glimpse at improving Theorem 1 by replacing 1.6383 with 2.

To avoid the recomputation stated above, one may wish simply to increase the value of T without resorting the zeroes. Unfortunately all this will accomplish is making the kernel closer to 1, meaning we will eventually hit a ceiling. For illustrative purposes, Figure 2.5 shows the value Theorem 1 could obtain if

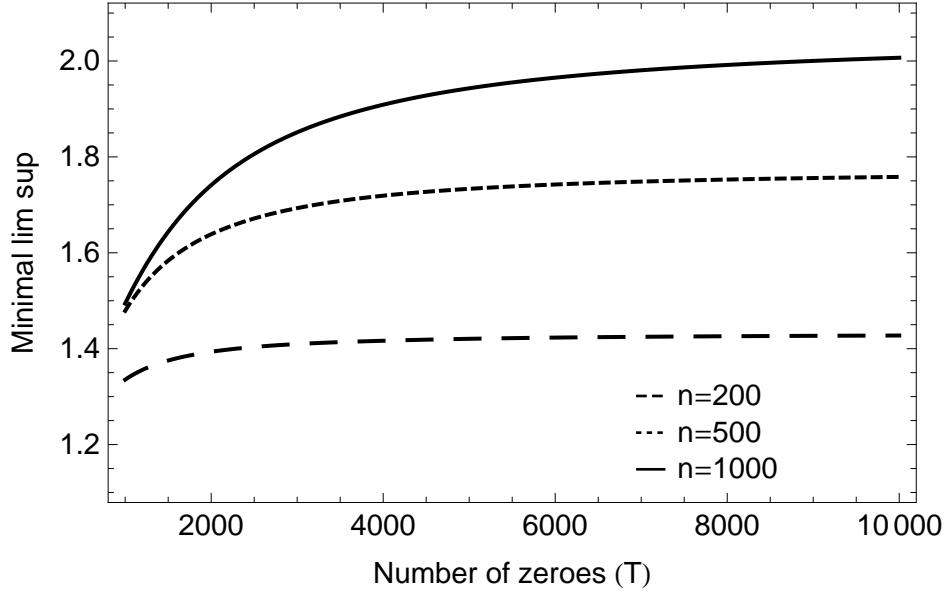


Figure 1: Smallest absolute value that the lim sup and lim inf can take if we use the first n zeroes sorted by using the appropriate T , assuming $\frac{2N_\gamma}{N_\gamma+1} \approx 2$.

one were to raise the value of T (assuming that the value of N_γ stays large). It uses the first 300 zeroes in sorted order (sorted using $T = \gamma_{2001} - \epsilon$), but T varies. We also include the values that the Féjer kernel would produce.

It is clear that the improved kernel approaches the maximal value quicker than the Féjer kernel. To obtain values within 0.01 of the optimal value, the Féjer kernel needs to check a total of 30398 zeroes, while the improved kernel only needs to check 6224 zeroes. When a higher precision is needed, the gap widens immensely: to obtain values within 0.001 of the optimal value, the Féjer kernel and the improved kernel need to check 399444 and 24043 zeroes, respectively.

There is also a trade-off when determining which value to take for δ in the LLL-reduction. As we took more zeroes, the differences really started to shine through. Taking larger values of δ yielded a slower program, but one that gave a much better value for N_γ . On the other hand, taking a smaller value of δ sped up the program, but gave much smaller values for N_γ .

Finally, we have assumed that each of the N_γ s be the same. This is not necessary for our method to work. The bound in Lemma 1 may be rewritten as

$$|v|^2 < 2 \max(N_\gamma) + 1 + \left(1 + \sum N_\gamma\right)^2 + \sum N_\gamma^2.$$

A similar bound may be drawn up for Lemma 2. These bounds, however, are only useful when the resulting value of N_γ from Theorem 3 is relatively small.

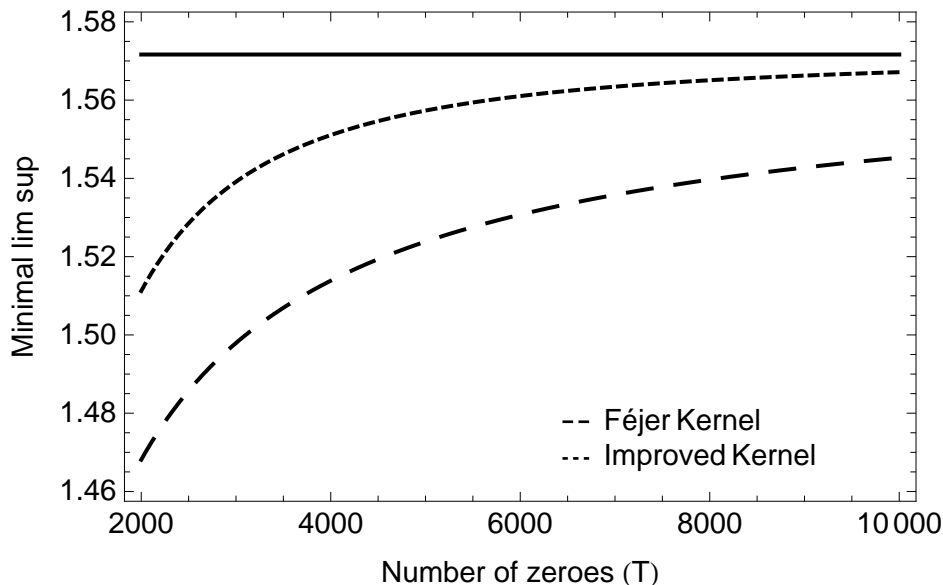


Figure 2: Smallest absolute value that the lim sup and lim inf can take if we use the first 300 zeroes sorted by using $T = \gamma_{2001} - \epsilon$ for the two different kernels. The horizontal line is the maximum attainable value for this set of zeroes.

2.6 Other theorems

In [6], two other important number-theoretic results are reproved on the condition that certain combinations of zeroes are linearly independent. Indeed, in [6], Theorem 1 gives us that $\pi(x) - \text{li}(x)$ changes sign infinitely often provided that the first 30 zeroes are 5-independent; Theorem 2 shows that the functions associated with conjectures of Pólya and Turán, respectively

$$L(x) = \sum_{1 \leq n \leq x} \lambda(n), \quad T(x) = \sum_{1 \leq n \leq x} \frac{\lambda(n)}{n},$$

where $\lambda(n)$ is the Liouville function, change sign infinitely often provided that the first 13 zeroes are 16-independent.

The data provided in Table 2 are more than enough to provide new proofs of these results.

A m -Independence

Table 1 gives the smallest sum (in absolute value) of the first N zeroes of the zeta function using coefficients $|c_n| \leq 1$. We have checked all permissible sums up to $N = 41$. The first 20 zeroes were checked in [2, Table I], labelled as *Type (A)*. They also provide a probabilistic value for the minimum value of such a sum, which we also include below.

To avoid a lengthy column of γ s to show the smallest linear combination, we encode the sums by an ordered pair of integers. If you write each integer in terms of its binary representation, a 1 in the i th least significant bit implies that γ_i is in the sum. The i th least significant bit being a 1 in the first (resp. second) coordinate gives us a positive (resp. negative) coefficient. For example, $(5, 24)$ represents the sum $\gamma_1 + \gamma_3 - \gamma_4 - \gamma_5$.

Table 1: Value of the Smallest Sums with Coefficients -1,0,1

N	Actual Value	Predicted Value	Linear Combination
20	2.9799×10^{-8}	1.3976×10^{-7}	(533185, 147768)
21	2.9799×10^{-8}	4.9104×10^{-8}	(533185, 147768)
22	2.9799×10^{-8}	1.7238×10^{-8}	(533185, 147768)
23	7.1672×10^{-9}	6.0341×10^{-9}	(3442980, 4273746)
24	1.1632×10^{-9}	2.1088×10^{-9}	(2626459, 12657764)
25	3.8873×10^{-10}	7.3493×10^{-10}	(17704982, 10589760)
26	1.0788×10^{-10}	2.5605×10^{-10}	(42549638, 3575905)
27	1.0788×10^{-10}	8.9049×10^{-11}	(42549638, 3575905)
28	1.8340×10^{-11}	3.0897×10^{-11}	(96882844, 171511617)
29	1.1519×10^{-11}	1.0713×10^{-11}	(93167683, 405819176)
30	9.1777×10^{-12}	3.7102×10^{-12}	(948312448, 41509390)
31	2.4115×10^{-12}	1.2836×10^{-12}	(1889619981, 88484592)
32	4.6939×10^{-14}	4.4343×10^{-13}	(2299561158, 1107850008)
33	4.6939×10^{-14}	1.5299×10^{-13}	(2299561158, 1107850008)
34	4.6939×10^{-14}	5.2784×10^{-14}	(2299561158, 1107850008)
35	1.8196×10^{-17}	1.8180×10^{-14}	(19757670928, 14533859426)
36	1.8196×10^{-17}	6.2574×10^{-15}	(19757670928, 14533859426)
37	1.8196×10^{-17}	2.1516×10^{-15}	(19757670928, 14533859426)
38	1.8196×10^{-17}	7.3945×10^{-16}	(19757670928, 14533859426)
39	1.8196×10^{-17}	2.5399×10^{-16}	(19757670928, 14533859426)
40	1.8196×10^{-17}	8.7157×10^{-17}	(19757670928, 14533859426)
41	1.8196×10^{-17}	2.9877×10^{-17}	(19757670928, 14533859426)

Table 2 gives us new lower bounds on the m -independence of the first n zeroes of the zeta function. By m -independence here, we mean that all non-trivial linear combinations of the first n zeroes are nonzero assuming the coefficients are no more than m in absolute value. This was computed using the same method as above, but keeping the zeroes in cardinal order. We include full results for the first 20 zeroes, and then only specific entries past there. Note that if k is not included in the table, any bound for the first $n > k$ zeroes also gives a lower bound for the first k zeroes.

Table 2: Linear independence of the first n zeroes of the zeta function

n	m	n	m
2	3.19683×10^{4499}	50	3.66786×10^{177}
3	7.01089×10^{2999}	75	6.96347×10^{116}
4	2.55333×10^{2249}	100	1.83869×10^{86}
5	3.18071×10^{1799}	125	4.96418×10^{67}
6	1.69018×10^{1499}	150	1.90667×10^{55}
7	6.90883×10^{1284}	175	1.35536×10^{46}
8	1.68884×10^{1124}	200	1.13717×10^{39}
9	1.12832×10^{999}	225	2.29079×10^{33}
10	1.21351×10^{899}	250	6.69056×10^{28}
11	1.33521×10^{817}	275	1.38130×10^{25}
12	9.26711×10^{748}	300	6.20938×10^{21}
13	1.57289×10^{691}	325	2.05342×10^{19}
14	5.10452×10^{641}	350	3.13279×10^{16}
15	7.35106×10^{598}	375	3.56683×10^{14}
16	1.51957×10^{561}	400	2.33172×10^{12}
17	1.34818×10^{528}	425	2.86453×10^{10}
18	5.05309×10^{498}	450	4.95180×10^8
19	1.74671×10^{472}	475	1.90299×10^7
20	3.58761×10^{448}	500	5.54632×10^5

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